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SUMMARY

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The effects of a constant external magnetic field on the laminar, fully developed flow of an electrically conducting incompressible rarefied gas in a nonconducting parallel-plate channel are studied. Consideration is given to the slip-flow regime, wherein a gas velocity discontinuity occurs at the channel walls. It is found that the magnitude of the slip velocity is unaffected by the magnetic-field strength for a given pressure drop, but that the mean gas velocity and wall friction coefficient are functions of both the velocity slip coefficient and the magnetic-field strength. The effect of a second-order slip-flow boundary condition is briefly discussed.

AUTHOR

INTRODUCTION

In recent years, considerable interest has developed in the study of the interaction between magnetic fields and the flow of electrically conducting fluids or gases. As is well known, gases at high temperatures become ionized; ionized gases conduct electricity and can be acted upon by magnetic fields. A growing body of both experimental and theoretical results on this subject has been obtained and has been reviewed in texts on magnetogasdynamics (e.g., ref. 1).

Another topic of current interest is the flow of low-density gases in channels. Gas flows under normal density conditions are called continuum flows. As the density is reduced, either through an increase in temperature or a decrease in absolute pressure, a departure from continuum gas-dynamics phenomena occurs. The first effect of the gas rarefaction is displayed as a slip of the gas over the bounding wall. This flow regime is termed slip flow. The flow of rarefied gases is conveniently summarized in ref. 2.

The study to be reported herein is concerned with the slip-flow regime. Specifically, the effects of a magnetic field on the flow of a slightly rarefied, conducting gas in a parallel-plate channel are studied. Within the knowledge of the author, the problem of slip flow in a magnetic field has received very little attention. Low-speed plane Couette flow of a rarefied conducting gas in a uniform transverse magnetic field has been considered (ref. 3) through equations developed from the Boltzmann equation of the kinetic theory of gases. The results are believed to be indicative of boundary-layer flow of a rarefied gas over a flat plate in the presence of a magnetic field.

The purpose of the present work is to study the combined effects of velocity slip and magnetic field on the steady laminar flow of an incompressible, electrically conducting gas of constant electrical conductivity between two parallel walls. The assumption of incompressible flow and uniform electrical conductivity in the entire flow field is physically realizable for the case of subsonic flows of a relatively hot gas. The velocity of the flow is parallel to the channel walls, and there is an external magnetic field of constant strength transverse to these walls. The more generally accepted method of analysis for slip flows is utilized here; i.e., the continuum equations of motion are used throughout the gas, together with the slip velocity boundary condition at the duct walls (ref. 2). It is felt that this study will yield some understanding of the interaction of a magnetic field with duct flow of a slightly rarefied conducting gas. Such flow problems could arise, for example, in magnetogasdynamic (MGD) generators that use relatively low-pressure, high-temperature gases as the working fluid, and in MGD space-flight propulsion systems.

In the next section, the first-order velocity slip boundary condition is imposed, and the results are considered in some detail. The effect of a second-order jump boundary condition is taken up briefly in the final section.

FIRST-ORDER SLIP FLOW

Dimensions and coordinates for the system under study are shown in fig. 1. The equation of motion for the fully-developed axial flow including a body force arising from the magnetic and electric fields is (ref. 4)

$$\partial p/\partial x = \mu_e J H_0 + \mu(d^2u/dz^2), \tag{1}$$

where $\partial p/\partial x$ is the axial pressure gradient, μ_e the gas permeability, j the current, H_o the imposed magnetic field, μ the gas viscosity, and u the axial gas velocity. The absolute cgs system of units is adopted here. From the generalized Ohm's law, there can be substituted (ref. 4)

$$j = \sigma(E - \mu_e u H_o), \tag{2}$$

where E is the electric-field intensity and σ is the gas electrical conductivity, to obtain the differential equation for u:

$$\partial p/\partial x = \mu_e \sigma H_o(E - \mu_e u H_o) + \mu(d^2u/dz^2). \tag{3}$$

Defining $\eta = z/L$ and $P = -\partial p/\partial x$ gives

$$d^2u/d\eta^2 - M^2 u = - M^2(P/\mu_e^2 \sigma H_o^2 + E/\mu_e H_o), \tag{4}$$

where M is a dimensionless parameter called the Hartmann number, $\mu_e H_o L(\sigma/\mu)^{1/2}$.

The slip-flow boundary condition that permits a slip velocity u_s at the duct walls ($\eta = \pm 1$) is written as (ref. 2)

$$u(\pm 1) \equiv u_s = \mp \left[(\xi_u/L)(du/d\eta)\right]_{\eta=\pm 1}. \tag{5}$$

The slip coefficient ξ_u is given by the expression (ref. 2)

$$\xi_u = \left[(2 - \beta)/\beta\right] \bar{l}, \tag{6}$$

where $\bar{l}$ is the mean free path, given by (ref. 2)

$$\bar{l} = \left(\sqrt{\pi/8}/0.499\right) \mu \left(\sqrt{RT}/p\right), \tag{7}$$

and β is termed Maxwell's reflection coefficient; R is the gas constant.

The solution for the velocity distribution satisfying eqs. (4) and (5) is

$$u(\eta) = (P/\mu_e^2\sigma H_o^2 + E/\mu_e H_o)(1 - \alpha \cosh M\eta/\cosh M), \tag{8}$$

where $\alpha \equiv 1 \Big/ \left[1 + (\xi_u/L)M \tanh M\right]$.

If conservation of current within any element of channel length is assumed; i.e., if there is no net current flow through the channel, then it follows that

$$\bar{j} = \frac{1}{2}\int_{-1}^{+1} j dz \equiv 0, \tag{9}$$

where $\bar{j}$ is the mean current. Substituting eqs. (2) and (8) into eq. (9) and carrying out the integration yield

$$(P/\mu_e^2\sigma H_o^2 + E/\mu_e H_o) = (P/\mu_e^2\sigma H_o^2)(M/\alpha \tanh M). \tag{10}$$

From the definition of the Hartmann number, however, $\mu_e^2\sigma H_o^2 = M^2\mu/L^2$, and hence an alternate form of eq. (10) is

$$(P/\mu_e^2\sigma H_o^2 + E/\mu_e H_o) = (PL^2/\mu)\Big/(\alpha M \tanh M). \tag{11}$$

Utilizing eq. (10) allows eq. (8) to become

$$u(\eta) = (PM/\mu_e^2\sigma H_o^2)\left[(\cosh M - \alpha \cosh M\eta)/\alpha \sinh M\right]. \tag{12}$$

The mean gas velocity $\bar{u}$ is obtained from the definition

$$\bar{u} = \frac{1}{2}\int_{-1}^{+1} u d\eta = (P/\mu_e^2 \sigma H_0^2)[(M \coth M/\alpha) - 1]. \tag{13}$$

With division of eq. (12) by eq. (13), a dimensionless velocity distribution in the duct is obtained:

$$u/\bar{u} = (1 - \alpha \cosh M\eta/\cosh M)/(1 - \alpha \tanh M/M). \tag{14}$$

This equation can be recast into an equivalent form that clearly reveals the significant dimensionless parameters:

$$u/\bar{u} = \left[\xi_u/L + (\cosh M - \cosh M\eta)/M \sinh M\right]$$

$$\div(\xi_u/L + 1/M \tanh M - 1/M^2)$$

$$\equiv \left[\lambda + g(M,\eta)\right] / \left[\lambda + f(M)\right], \tag{15}$$

where $\lambda \equiv \xi_u/L$. It is seen that the velocity profiles thus generated depend on the functions $f(M)$ and $g(M)$, which are functions of the parameter M only, and on the parameter λ. Fig. 2 shows typical velocity distributions for laminar flow in a parallel plate channel under a transverse magnetic field of different strengths.

Before proceeding further, it is illuminating to determine the slip velocity u_s. Combining eqs. (8) and (11) gives

$$u(\eta) = \left[(PL^2/\mu)/(\alpha M \tanh M)\right] (1 - \alpha \cosh M\eta/\cosh M). \tag{16}$$

Hence, the velocity gradient at the upper wall $\eta = 1$ is

$$(du/d\eta)_{\eta=1} = -\left[(PL^2/\mu)/(\alpha M \tanh M)\right](\alpha M \tanh M) = -PL^2/\mu. \tag{17}$$

It is seen that this gradient is independent of the parameters M and λ. Thus gas rarefaction and/or imposed magnetic field apparently have no effect on the velocity gradient at the wall.[1] Now, the slip velocity u_s is given by

$$u_s = -\left[(\xi_u/L)(du/d\eta)\right]_{\eta=1} = (\xi_u/L)(PL^2/\mu). \tag{18}$$

From eq. (18), it is easily seen that the magnitude of the slip velocity is only dependent on the parameter ξ_u/L (for a given pressure drop) and is not a function of the magnetic-field strength. This has important mathematical and physical consequences, as will be shown later.

From eq. (15), the dimensionless slip velocity $u_s/\bar{u}$ and the centerline velocity $u_c/\bar{u}$ are obtained by setting $\eta = 1$ and $\eta = 0$, respectively:

$$u_s/\bar{u} = \lambda/\left[\lambda + f(M)\right]; \tag{19}$$

$$u_c/\bar{u} = \left[\lambda + \tanh(M/2)/M\right]/\left[\lambda + f(M)\right] = \left[\lambda + g(M)\right]/\left[\lambda + f(M)\right]. \tag{20}$$

It is of interest to examine the velocity profiles for the limiting cases $M = 0$ and large M. For $M = 0$, it can easily be shown

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[1] This result is not mentioned in ref. 4 for continuum flow (λ=0) although fig. 1 of this reference suggests it was known to the author.

that $f(0) = 1/3$ and $g(0) = 1/2$, so that

$$\left.\begin{aligned} u/\bar{u} &= \left[(3/2)(1 - \eta^2) + 3\lambda\right]/(1 + 3\lambda), \\ u_s/\bar{u} &= 3\lambda/(1 + 3\lambda), \\ u_c/\bar{u} &= (3/2 + 3\lambda)/(1 + 3\lambda), \end{aligned}\right\} \quad M = 0 \tag{21}$$

which are the solutions for fully developed slip flow in a parallel plate channel in the absence of a magnetic field(ref. 5). For large M, $f(M) \to 0$, $g(M) \to 0$, and consequently

$$\left.\begin{aligned} u_s/\bar{u} &\to \lambda/\lambda = 1, \\ u_c/\bar{u} &\to \lambda/\lambda = 1, \\ u_s &= \lambda(PL^2/\mu). \end{aligned}\right\} \quad \text{large } M \tag{22}$$

Hence, from eq. (22), one would conclude that for large M

$$\bar{u} \to u_c \to u_s = \lambda\, PL^2/\mu \tag{23}$$

so that the gas velocity is uniform across the duct and equal to zero only in the absence of gas rarefaction. Since this is not a physically plausible result, the foregoing results must be restricted to small values of the Hartmann number M.

Equations for $u_s/\bar{u}$ (eq. (19)) and $u_c/\bar{u}$ (eq. (20)) have been evaluated as functions of the two parameters M and λ and are plotted in figs. 3 and 4, respectively. Fig. 3 gives the velocity slip at the wall. It is seen that the dimensionless slip velocity $u_s/\bar{u}$ increases toward unity as M increases. Fig. 4 shows the behavior of the dimensionless centerline velocity $u_c/\bar{u}$. In the absence of a magnetic field $M = 0$, the effect of slip decreases this ratio from its maximum value of 1.5 (continuum flow) toward a value of 1.0. If a magnetic field is imposed, this decrease is accentuated with increasing magnetic-field strength.

The current distribution j is found from eq. (2), which gives j in terms of the imposed uniform electric and magnetic fields and the velocity u

$$j = \sigma(E - \mu_e u H_o)$$

$$= (P/\mu_e H_o)\left[(M \cosh M\eta/\sinh M) - 1\right] \tag{24}$$

when eqs. (9), (10), and (12) are used. Hence, as is perhaps expected, the current distribution is not dependent on the slip parameter λ.

As a matter of general interest, an examination shall be made as to how the wall friction is affected by the gas rarefaction and imposed magnetic field. A wall friction coefficient for laminar flow is defined as

$$S = \tau_\omega L/\mu\bar{u}, \tag{25}$$

where τ_ω is the shear stress at the wall:

$$\tau_\omega = -\mu(du/dz)_{z=L} = -\left[(\mu/L)(du/d\eta)\right]_{\eta=1}. \tag{26}$$

Evaluating the wall velocity gradient from eq. (5) results in

$$S = (1/\lambda)(u_s/\bar{u}) = 1/\left[\lambda + f(M)\right] \tag{27}$$

with eq. (19).

In the absence of rarefaction effects, the value of S is 3 for zero magnetic-field strength and increases without limit as the magnetic-field strength increases. If, however, the gas is slightly rarefied, the friction coefficient is reduced, for a given value of M, below the corresponding value for continuum flow. It is interesting to note that eq. (27) predicts a limiting value for S of $1/\lambda$ as M increases indefinitely. These results are shown in fig. 5.

The relation between the mean velocity $\bar{u}$ and the pressure drop P for the flow is of considerable interest and can be obtained readily from eq. (13):

$$\bar{u} = (P/\mu_e^2\sigma H_o^2)\left[M \coth M/\alpha) - 1\right]. \tag{28}$$

This equation can be recast into the more illuminating form

$$\bar{u} = (PL^2/\mu)\left[\lambda + f(M)\right] \equiv (PL^2/3\mu)\psi, \tag{29}$$

where $\psi \equiv 3(\lambda + f)$.

In the absence of gas rarefaction and imposed magnetic field, the correction factor ψ has the value of 1.0:

$$\bar{u}\Big|_{\substack{\lambda=0\\ M=0}} = PL^2/3\mu. \tag{30}$$

With gas rarefaction and/or applied magnetic field, the resulting flow rate, for a given pressure drop, is ψ times the value predicted by eq. (30). The correction factor ψ has been evaluated as a function of the two parameters $\mu\sqrt{RT}/\overset{p}{\cancel{P}}L$ and M, and is plotted in fig. 6 for $\beta = 1$. Gas rarefaction tends to increase the flow rate, for a given pressure drop, over that predicted by eq. (30), while the presence of a magnetic field tends to suppress the flow rate. The direction of these results is physically reasonable since with gas rarefaction there is a lessening of frictional resistance to flow (fig. 5), while the presence of a magnetic field introduces a term in the momentum equation having a component always opposite to the direction of gas flow (hence a force that retards the flow rate). The correction factor ψ becomes inaccurate as $M \to \infty$, for then

$$\psi\Big|_{M \to \infty} = 3\xi_u/L, \tag{31}$$

which would suggest that only in the absence of gas rarefaction can the magnetic field completely suppress the flow through the duct.

SECOND-ORDER SLIP FLOW

The results of the preceding section are based on a first-order slip-flow boundary condition. In this section, brief consideration will be given to the effects of a second-order boundary condition. The second-order slip-flow boundary condition is believed applicable at somewhat lower pressures (or higher temperatures) than is the first-order condition.

The differential equation governing the flow velocity is, from eq. (4),

$$d^2u/d\eta^2 - M^2u = -M^2(P/\mu_e^2\sigma H_o^2 + E/\mu_e H_o). \tag{32}$$

The slip-flow boundary condition in the present analysis is (ref. 6)

$$u(\pm 1) = u_s = \mp \lambda(du/d\eta)_{\eta=\pm 1} + \lambda^*(d^2u/d\eta^2)_{\eta=\pm 1}, \tag{33}$$

where (ref. 6)

$$\left.\begin{aligned} \lambda &= \xi_u/L = \sqrt{\frac{\pi}{2}}\left[(2-\beta)/\beta\right](\mu\sqrt{RT}/pL) \\ \lambda^* &= -(9\pi/16)(\mu\sqrt{RT}/pL) \end{aligned}\right\} \tag{34}$$

The solution for the velocity distribution u is found to be

$$u(\eta) = (PL^2/\mu)(1/M)\left[(\cosh M - \alpha^* \cosh M\eta)/\alpha^* \sinh M\right], \tag{35}$$

where $\alpha^* = 1/(1 + \lambda M \tanh M - \lambda^* M^2)$. Setting $\eta = 1$ obtains the slip velocity u_s:

$$\begin{aligned} u_s &= (PL^2/\mu)(1/M)\left[(1 - \alpha^*)/\alpha^* \tanh M\right] \\ &= (PL^2/\mu)(1/M)\left[(\lambda M \tanh M - \lambda^* M^2)/\tanh M\right] \\ &= (PL^2/\mu)(\lambda - \lambda^* M/\tanh M). \end{aligned} \tag{36}$$

It is seen from eq. (36) that consideration of a second-order boundary condition introduces an additional term $\lambda^* M/\tanh M$ in the slip velocity expression. Now, for $M \to 0$ (no magnetic field), $M/\tanh M \to 1$, and hence

$$u_s\big|_{M=0} = (PL^2/\mu)(\lambda - \lambda^*), \tag{37}$$

which is correct (ref. 6). On the other hand, as M increases, $\tanh M \to 1$, and consequently the term $\lambda^*(M/\tanh M)$ increases without limit. Thus, u_s becomes indefinitely large as M increases. This result implies that a second order analysis is no longer adequate as M increases in magnitude. As a consequence, it is concluded that the use of a second-order slip boundary condition, as well as the first-order boundary condition, is restricted to low values of the Hartmann number M.

For the sake of completeness, expressions for the wall friction coefficient S and mean velocity $\bar{u}$ are as follows:

$$S = 1/\left[\lambda - \lambda^* M/\tanh M + f(M)\right], \tag{38}$$

$$\bar{u} = 3PL^2/3\mu\left[\lambda - \lambda^* M/\tanh M + f(M)\right], \tag{39}$$

where, again,

$$f(M) = 1/M \tanh M - 1/M^2. \tag{40}$$

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Figure 4. - Centerline-to-mean velocity ratio $U_c/\overline{U}$ vs. Hartmann number M and slip parameter λ.

Figure 5. - Wall friction coefficient S vs. Hartmann number M and slip parameter λ.

Figure 6. - Flow rate correction factor ψ vs. Hartmann number M and slip parameter λ.

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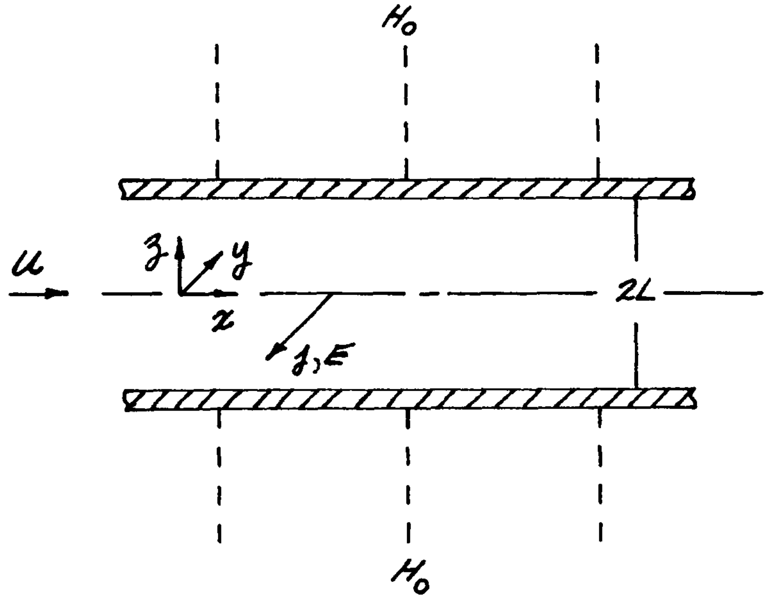

Figure 1

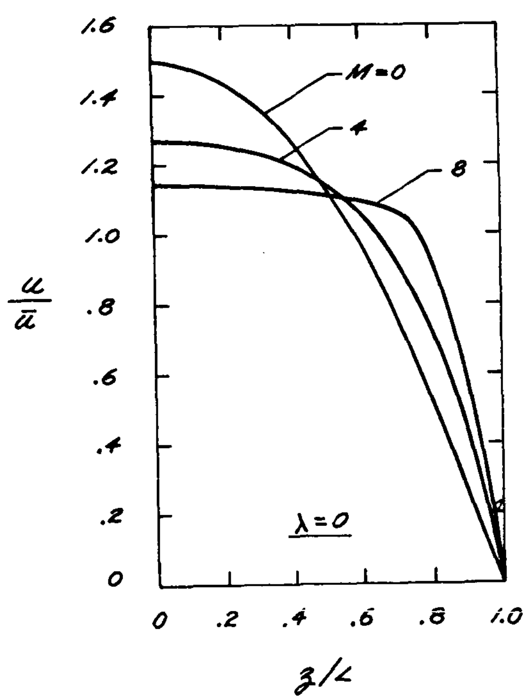

Figure 2a

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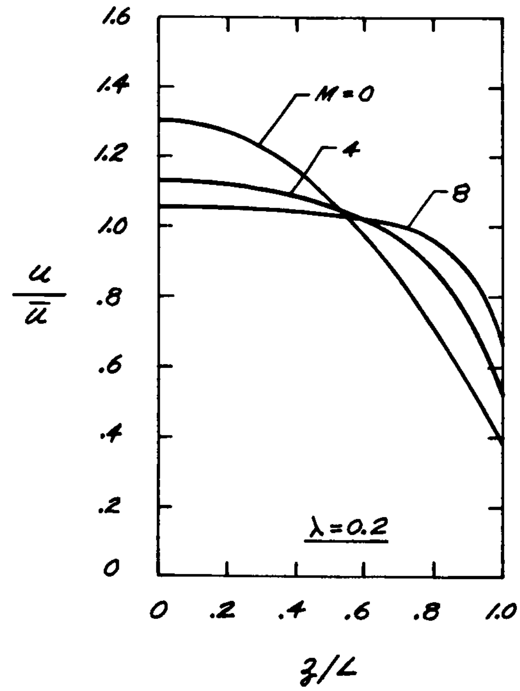

Figure 2b

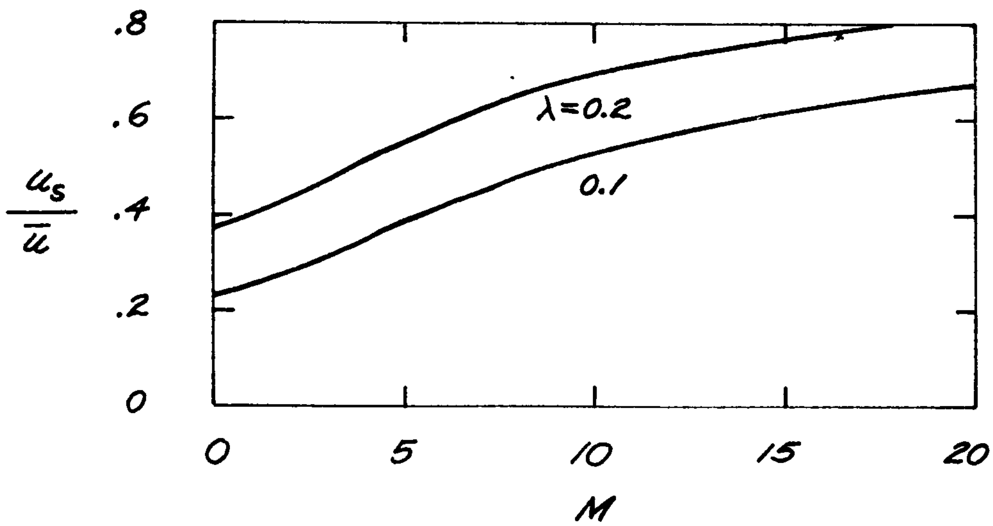

Figure 3

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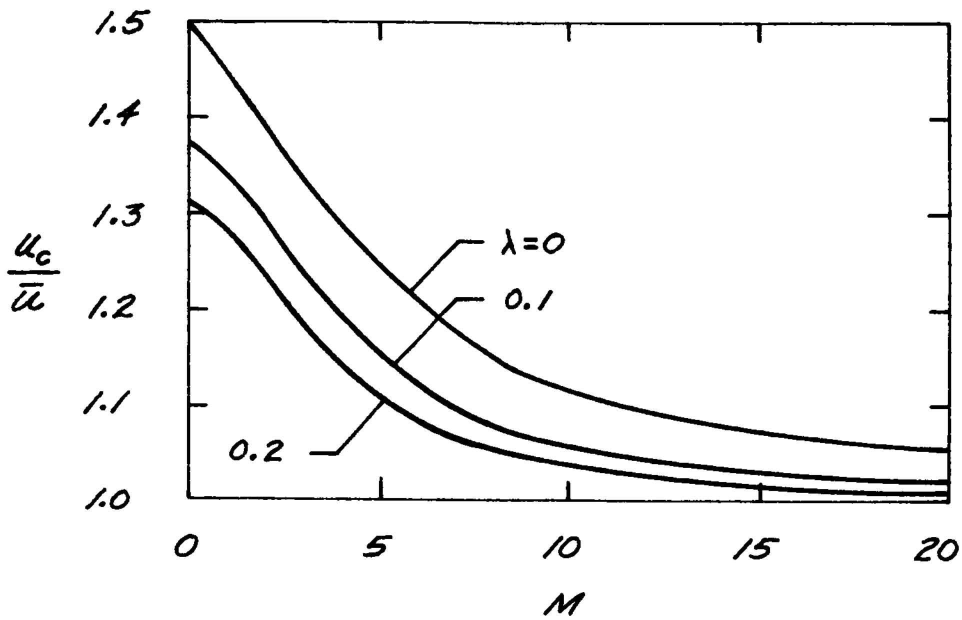

Figure 4

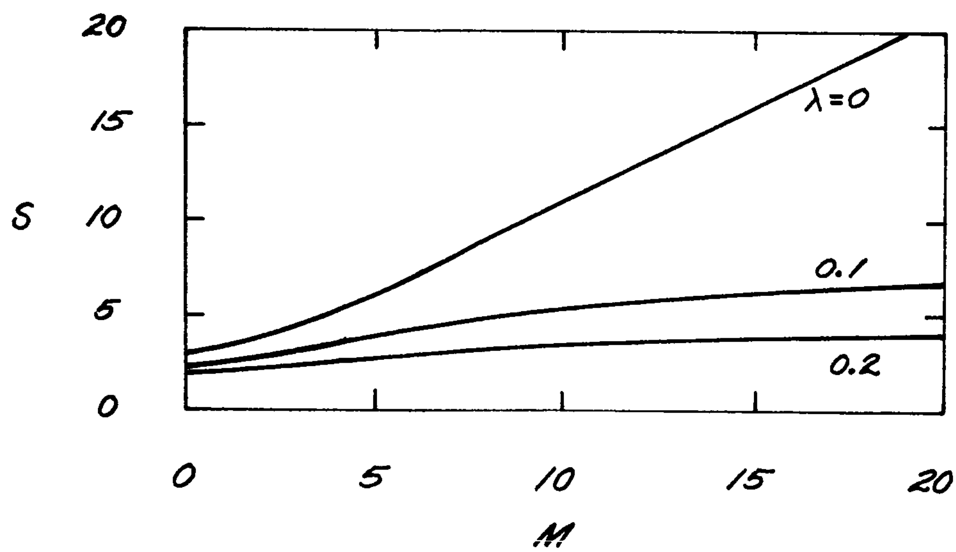

Figure 5

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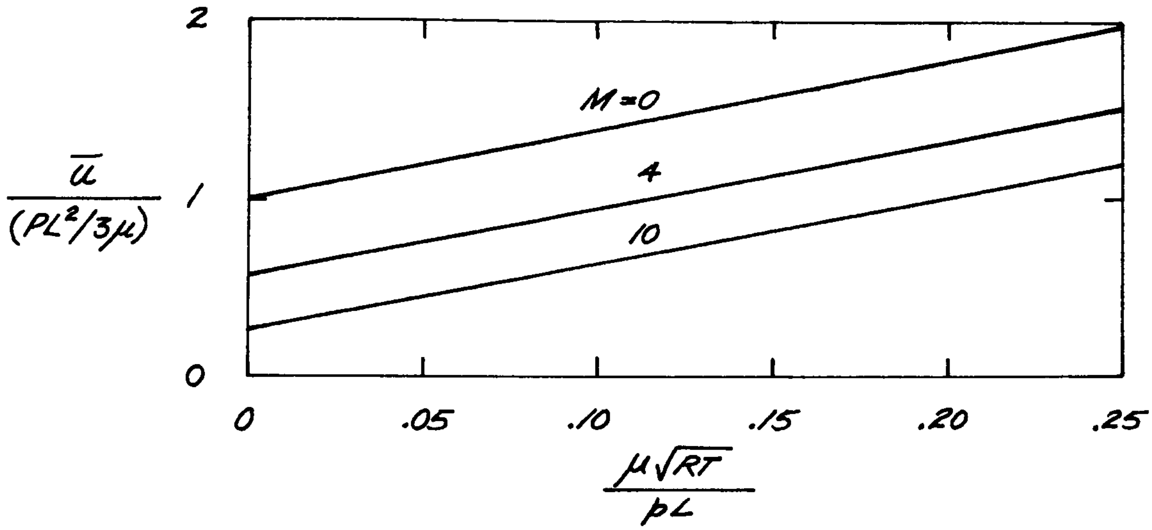

Figure 6

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